South Australian
Certificate of Education

1

Specialist Mathematics
2021

Question booklet 1

Questions 1 to 7 (55 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 16 if you need more space
- Allow approximately 70 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams

Total time: 130 minutes
Total marks: 100

Attach your SACE registration number label here

Graphics calculator

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Question 1 (6 marks)

(a) Write $-1+i\sqrt{3}$ in $r\,\mathrm{cis}\theta$ form.

(1 mark)

(b) Consider the complex number $z_1 = x + iy$, where $x > 0$, $y > 0$, and $x > y$.

The complex number z_1, which lies in the first quadrant of the Argand diagram, is shown in Figure 1.

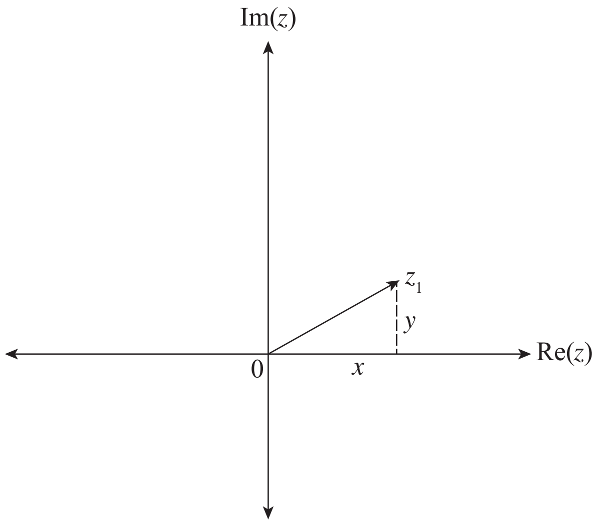

Figure 1

(i) Let $z_2 = (-1+i\sqrt{3})z_1$.

Using part (a), show that $|z_2| = 2|z_1|$.

(1 mark)

(ii) On the Argand diagram in Figure 1, draw z_2. (2 marks)

(c) Use the triangle inequality to show that $|z_1 - z_2| < 3|z_1|$.

(2 marks)

Question 2 (6 marks)

Figure 2 shows a diagram of an elliptical-shaped oil spill that is expanding in area on the ocean surface.

The area of an ellipse is $A = \pi ab,$ where a and b are measurements on the axes of symmetry, as shown in Figure 2.

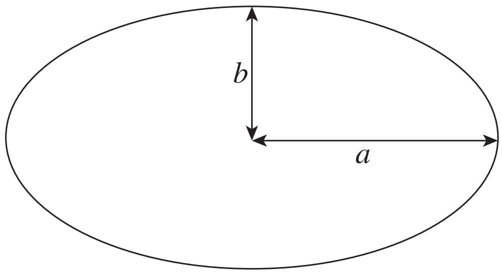

Figure 2

The rate of change of the area of the elliptical oil spill is given by $\frac{dA}{dt}$. It may be assumed that the elliptical shape is maintained as the oil spill expands.

(a) Show that $a\frac{db}{dt} = \frac{1}{\pi}\frac{dA}{dt} - b\frac{da}{dt}$.

(2 marks)

(b) Consider the instant when the area A is $12\,\text{m}^2$.

(i) Show that $a = \frac{12}{\pi b}$.

(1 mark)

(ii) The area of the oil spill is expanding at a rate of $2\,\text{m}^2\,\text{s}^{-1}$ at the instant when

$$A = 12\,\text{m}^2,\ b = 2\,\text{m},\ \text{and}\ \frac{\mathrm{d}a}{\mathrm{d}t} = 0.5\,\text{m}\,\text{s}^{-1}.$$

Find the ***exact*** value of $\frac{\mathrm{d}b}{\mathrm{d}t}$ at this instant.

(3 marks)

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Question 3 (6 marks)

Let $A = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & 2 \end{bmatrix}$.

(a) Use mathematical induction to prove that $A^n = \begin{bmatrix} \left(\frac{1}{3}\right)^{2n} & 0 \\ 0 & 2^n \end{bmatrix}$ for all positive integers n.

(5 marks)

(b) Using part (a), find the positive integer n such that $A^n \begin{bmatrix} 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 2^{2021} \end{bmatrix}$.

(1 mark)

Question 4 (10 marks)

Consider the function $f(x) = \dfrac{x^3 - 2x + 5}{x^2 + 1}$.

(a) Use a division process to show that $f(x) = x - \dfrac{3x - 5}{x^2 + 1}$.

(2 marks)

(b) On the axes in Figure 3, draw the function $f(x) = x - \dfrac{3x - 5}{x^2 + 1}$.

Clearly show the behaviour of the function near any asymptotes.

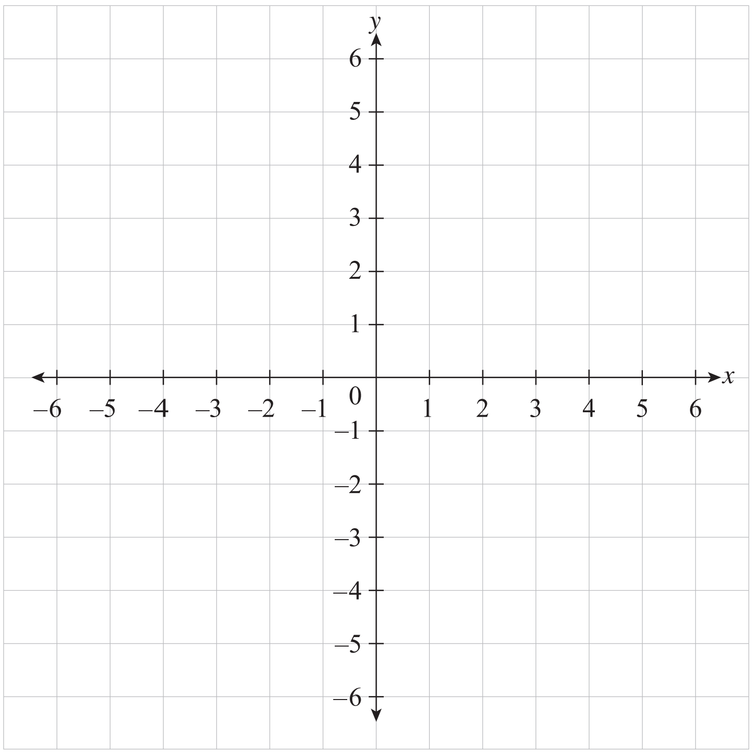

Figure 3

(3 marks)

(c) Find $g(f(x))$, given $g(x)=\sqrt{x}$ and $f(x)=x-\dfrac{3x-5}{x^2+1}$.

(1 mark)

The graph of $y=g(f(x))$ is shown in Figure 4.

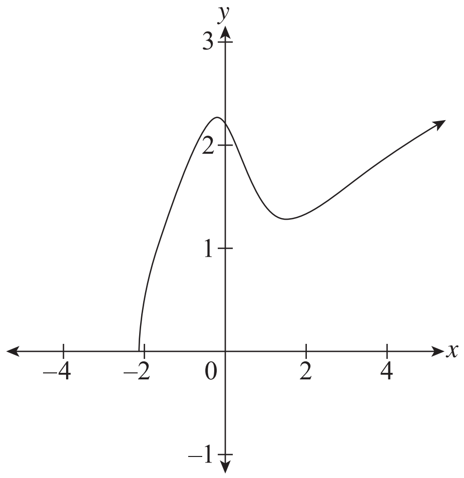

Figure 4

(d) Consider the solid obtained by rotating the graph of $y=g(f(x))$ about the x-axis between $x=-1$ and $x=1$.

(i) Show that the volume of this solid is given by the equation

$$V=\pi\int_{-1}^{1}\left(x-\frac{3x}{x^2+1}+\frac{5}{x^2+1}\right)\mathrm{d}x.$$

(1 mark)

(ii) Show that the ***exact*** volume of this solid is $\frac{5\pi^2}{2}$.

(3 marks)

Question 5 (6 marks)

Consider the following system of equations where m is a non-zero real number.

$$x+y=0$$
$$mx+z=m^2-1$$
$$mx+2my+\left(3-m^2\right)z=0$$

(a) Write this system of equations as an augmented matrix.

(1 mark)

(b) Using clearly stated row operations, show that the system in part (a) reduces to:

$$\begin{bmatrix} 1 & 1 & 0 & : & 0 \\ 0 & m & -1 & : & \left(1-m^2\right) \\ 0 & 0 & \left(m^2-4\right) & : & \left(1-m^2\right) \end{bmatrix}.$$

(c) (i) State a value of m for which there is a unique solution.

(1 mark)

(ii) Which figure below best represents the solution to this system for $m = -2$?

(1 mark)

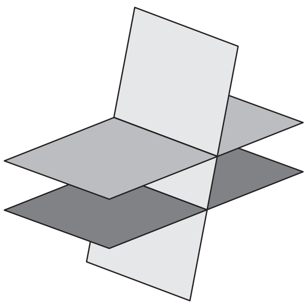

Figure A

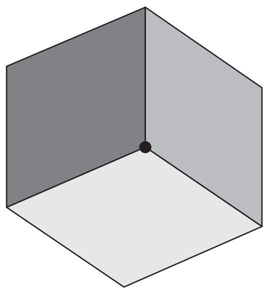

Figure B

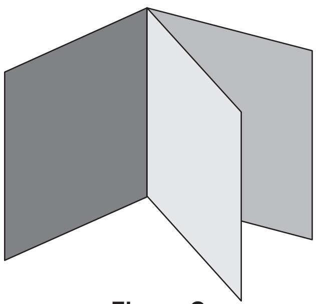

Figure C

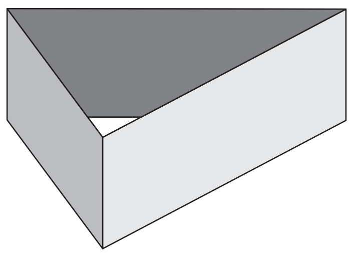

Figure D

Question 6 (10 marks)

(a) Use integration by parts to show that $\int \arccos x \, dx = x \arccos x - \sqrt{1-x^2} + c$.

(2 marks)

(b) (i) On the axes in Figure 5, draw and label the graph of $f(x) = \arccos x - \frac{\pi}{2}$ for $-1 \le x \le 1$.

(2 marks)

(ii) On the axes in Figure 5, draw and label the graph of $y = |f(x)|$ for $-1 \le x \le 1$. (1 mark)

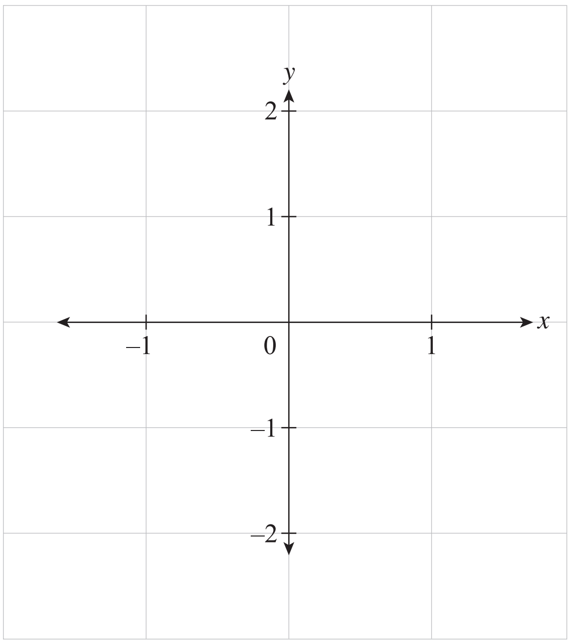

Figure 5

(iii) On the axes in Figure 6, draw the graph of $y = f(|x|)$ for $-1 \le x \le 1$.

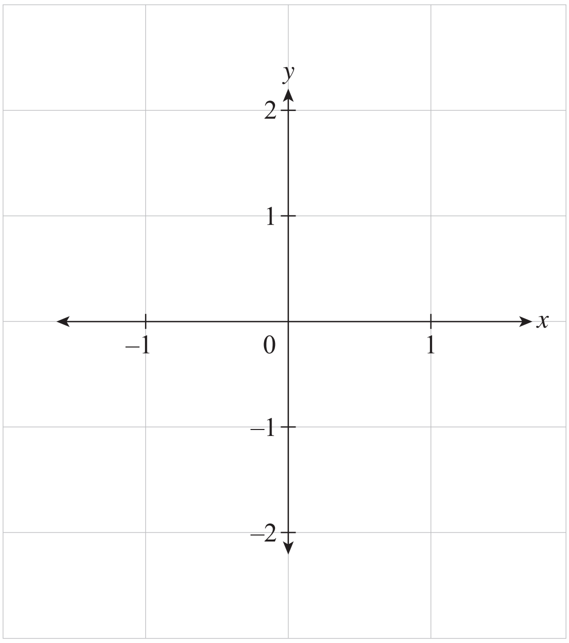

Figure 6

(1 mark)

(c) Using part (a) and part (b)(iii), show that the area between the graph of $y = f(|x|)$ and the y-axis for $0 \le x \le 1$ is 1 square unit.

(4 marks)

Question 7 (11 marks)

A species of animal is in danger of extinction.

The rate of change of the population is given by $\frac{\mathrm{d}y}{\mathrm{d}t} = -0.1y\left(\frac{T-y}{T}\right)$, where t is time in years after the population, y, was initially counted, and T is a constant.

The constant T is the threshold level of the population. The species will become extinct if the population falls below T.

(a) Show that $\frac{T}{y(T-y)} = \frac{1}{y} + \frac{1}{T-y}$.

(1 mark)

(b) Initially there were $20\,000$ animals, that is, $y(0) = 20\,000$.

(i) Using integration, show that the population of the animals is given by

$$y = \frac{20\,000T}{(T-20\,000)e^{0.1t} + 20\,000}.$$

(5 marks)

(ii) Find, to the nearest whole number, the value of the threshold level, T, if 50 years after the population was initially counted there were 5000 animals.

(2 marks)

(iii) Figure 7 shows the slope field of the solutions to the differential equation for the value of T found in part (b)(ii).

On the slope field in Figure 7, draw the solution curve using the initial condition and the information given in part (b)(ii).

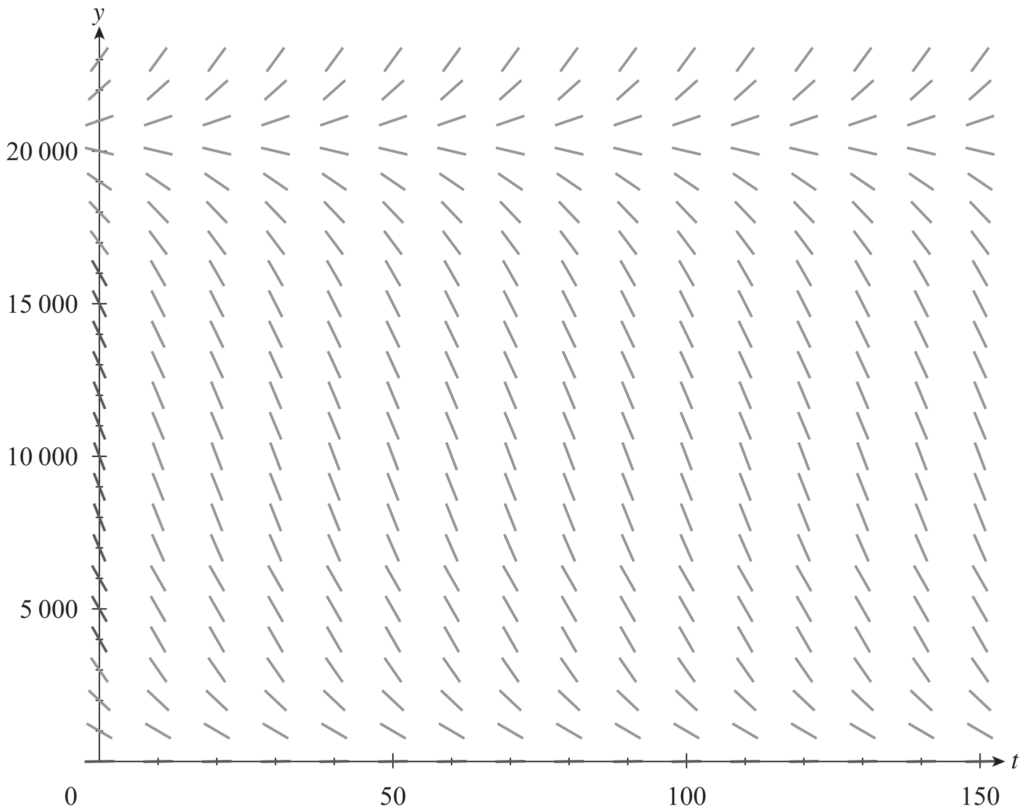

Figure 7

(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 7(b)(i) continued).

South Australian Certificate of Education

Specialist Mathematics
2021

Question booklet 2

Questions 8 to 10 (45 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 5 and 11 if you need more space
- Allow approximately 60 minutes
- Approved calculators may be used — complete the box below

2

Copy the information from your SACE label here

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Graphics calculator

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Model ____________

2. Brand ____________

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Question 8 (15 marks)

Points $A(5,-1,-3)$, $B(5,-3,-1)$, and $D(1,-1,1)$ are on the circumference of a circle with centre $C(3,-1,-1)$ on the plane P_1, as shown in Figure 8.

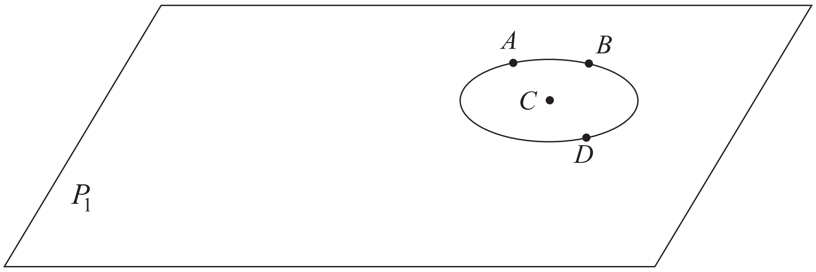


Figure 8

(a) (i) Find $\overrightarrow{BA} \times \overrightarrow{BD}$.

(2 marks)

(ii) Hence show that the equation of plane P_1 is $x + y + z = 1$.

(2 marks)

(b) (i) Show that AD is a diameter of the circle.

(1 mark)

(ii) Find the radius of the circle.

(1 mark)

(c) Point $E(8,-4,-3)$ is on the plane P_1.

Show that the parametric equations of the line through E and B are:

$$\begin{cases} x = 8 - 3t \\ y = -4 + t \\ z = -3 + 2t \end{cases} \quad \text{where } t \text{ is a real parameter.}$$

(2 marks)

(d) The equation of the circle on P_1 with centre C and passing through A, B, and D is:

$$(x-3)^2 + (y+1)^2 + (z+1)^2 = 8.$$

Show that the line through E and B intersects the circle again at $X\left(\frac{11}{7}, -\frac{13}{7}, \frac{9}{7}\right)$.

(4 marks)

(e) Find the arc length BX.

(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 8(a)(i) continued).

Question 9 (14 marks)

A planet is positioned at the point $P(0,0)$. The path of a comet approaching this planet is represented by the parametric equations

$$\begin{cases} x = 2\sec t - 1 \\ y = \sqrt{3}\tan t \end{cases} \quad \text{where } t \text{ is time.}$$

The path of the comet is observed for $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

(a) (i) On the axes in Figure 9, sketch the path of the comet.

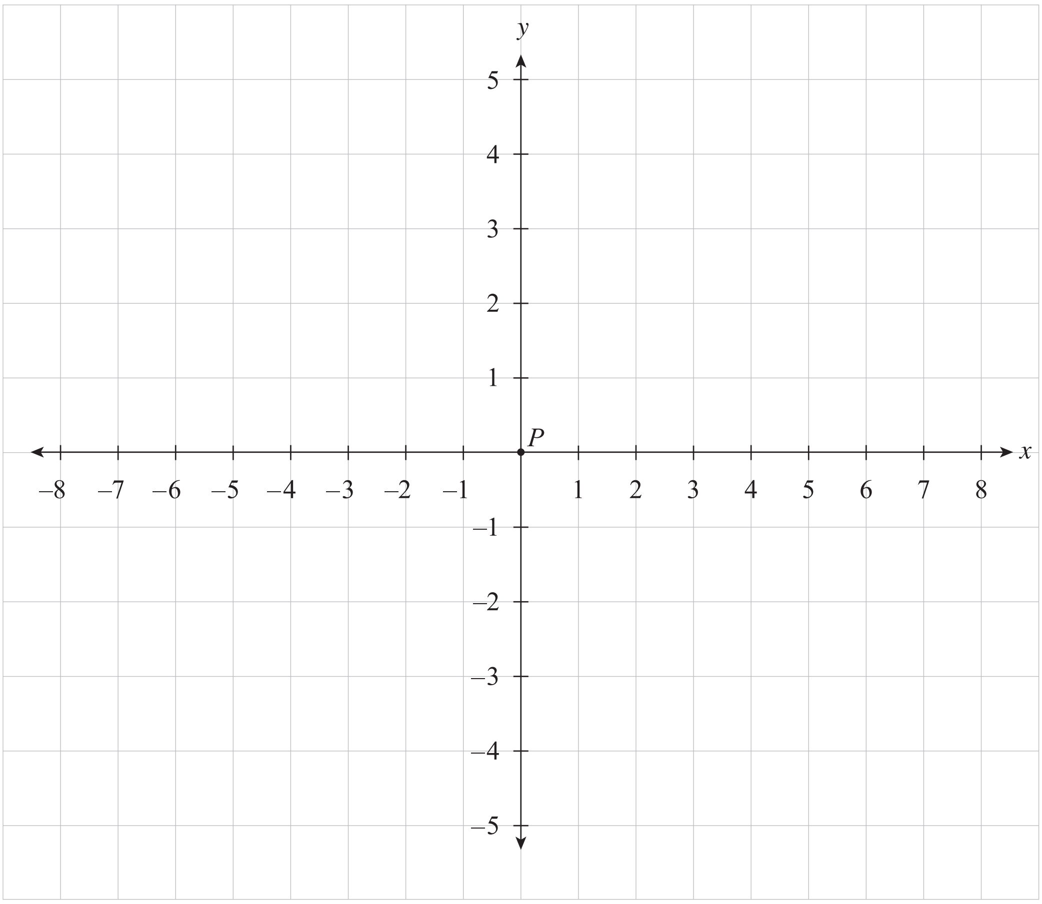


Figure 9

(3 marks)

(ii) At what time is the comet closest to the planet?

(1 mark)

(iii) On the axes in Figure 9, mark as point A the position of the comet at the time found in part (a)(ii). (1 mark)

(b) (i) Show that $\frac{dy}{dx} = \frac{\sqrt{3}}{2\sin t}$.

(3 marks)

(ii) For $t = \frac{5\pi}{6}$, find the ***exact*** value of $\frac{dy}{dx}$.

(1 mark)

(iii) On the axes in Figure 9, show the result of part (b)(ii). (1 mark)

(c) (i) Show that the speed, S, of the comet at time t is given by $S = \frac{\sqrt{4\sin^2 t + 3}}{\cos^2 t}$.

(2 marks)

(ii) Hence find the distance travelled by the comet for $\frac{2\pi}{3} \le t \le \frac{5\pi}{6}$.

(2 marks)

Question 10 (16 marks)

(a) (i) State the roots of the complex equation $w^6 = 1$ in $r\,\mathrm{cis}\,\theta$ form.

(2 marks)

(ii) On the Argand diagram in Figure 10, plot the roots identified in part (a)(i).

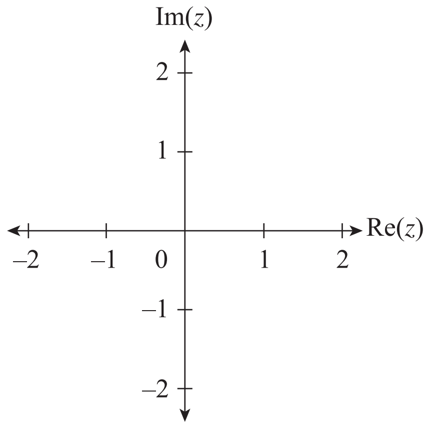


Figure 10

(2 marks)

(b) Consider the complex equation $(z-1)^6 = 1$.

(i) Using the roots identified in part (a)(i), state all the roots of the equation $(z-1)^6 = 1$, giving answers in any form.

(2 marks)

(ii) On the Argand diagram in Figure 11, plot the roots of the equation $(z-1)^6 = 1$.

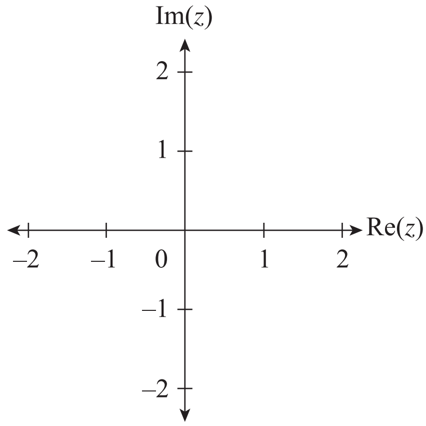


Figure 11

(1 mark)

(iii) Write the roots of $(z-1)^6 = 1$ in $r\,\text{cis}\,\theta$ form or real form.

(2 marks)

(c) (i) Suppose that the polynomial $z^2 + bz + c$ has a zero $r\,\text{cis}\,\theta$, where b and c are real, and $r > 0$ and $0 < \theta < \pi$.

Show that $b = -2r\cos\theta$ and $c = r^2$.

(2 marks)

(ii) Verify that $(z-1)^6 = z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z + 1$.

(2 marks)

(d) Using part (b)(iii) and part (c), factorise $z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z$ into the product of real linear and real quadratic factors.

(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 9(c)(ii) continued).

SPECIALIST MATHEMATICS FORMULA SHEET

Circular functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$\sin A \pm \sin B = 2 \sin \frac{1}{2}(A \pm B) \cos \frac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$

$\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$

Matrices and determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

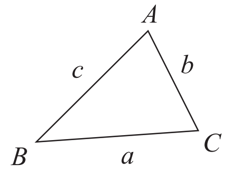
$A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Measurement

Area of sector, $A = \frac{1}{2} r^2 \theta$, where θ is in radians.

Arc length, $l = r\theta$, where θ is in radians.

In any triangle ABC:



Area of triangle $= \frac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a point to a plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
$\arcsin x$	$\dfrac{1}{\sqrt{1 - x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1 - x^2}}$
$\arctan x$	$\dfrac{1}{1 + x^2}$

Properties of derivatives

$\dfrac{d}{dx}\big(f(x)\,g(x)\big) = f'(x)g(x) + f(x)g'(x)$

$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$

$\dfrac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$

Arc length along a parametric curve

$l = \displaystyle\int_a^b \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}\, dt$, where $a \leq t \leq b$.

Integration by parts

$\displaystyle\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$

Volumes of revolution

About x axis, $V = \displaystyle\int_a^b \pi y^2 dx$, where y is a function of x.

About y axis, $V = \displaystyle\int_c^d \pi x^2 dy$, where y is a one-to-one function of x.