Stage 2 Specialist Mathematics

Sample examination questions - 3

Question 1 (5 marks)

Consider the graph of the relation $y^2 = 2\sin 2x + 2$ for $-\frac{\pi}{2} \le x \le \pi$, as shown in Figure 1.

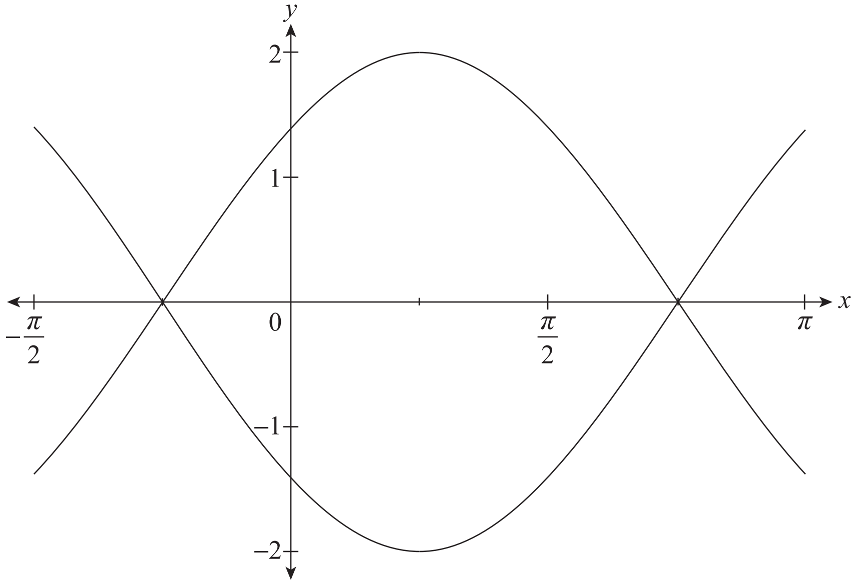


Figure 1

(a) Using implicit differentiation, show that $\frac{dy}{dx} = \frac{2\cos 2x}{y}$ when $y \neq 0$.

(3 marks)

(b) Find the *exact* values of $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{2}$.

(2 marks)

Question 2 (8 marks)

Consider the quartic function $P(x) = x^4 - bx^3 - x^2 + ax - (a-3)$ where a and b are real constants.

(a) (i) The polynomial $P(x)$ has one complex zero of $2+i$.

Find *one other* complex zero of $P(x)$.

(1 mark)

(ii) Hence find a real quadratic factor of $P(x)$.

(1 mark)

(b) (i) When $P(x)$ is divided by $(x+1)$, the remainder is -20.

Show that $b - 2a = -23$.

(2 marks)

(ii) When $P(x)$ is divided by $(x-2)$, the remainder is 4.

Show that $8b-a=11$.

(1 mark)

(iii) Find the values of a and b.

(1 mark)

(c) (i) Find $P(x)$ as a product of two real quadratic factors.

(1 mark)

(ii) State all the zeros of $P(x)$.

(1 mark)

Question 3 (8 marks)

The sets of complex numbers A, B, and C are described by the equations below.

$$A : |z-1| = 1$$

$$B : \operatorname{Im} z = \frac{1}{2}$$

$$C : \arg z = \frac{\pi}{4}$$

(a) On the Argand diagram in Figure 2, sketch and label sets A, B, and C.

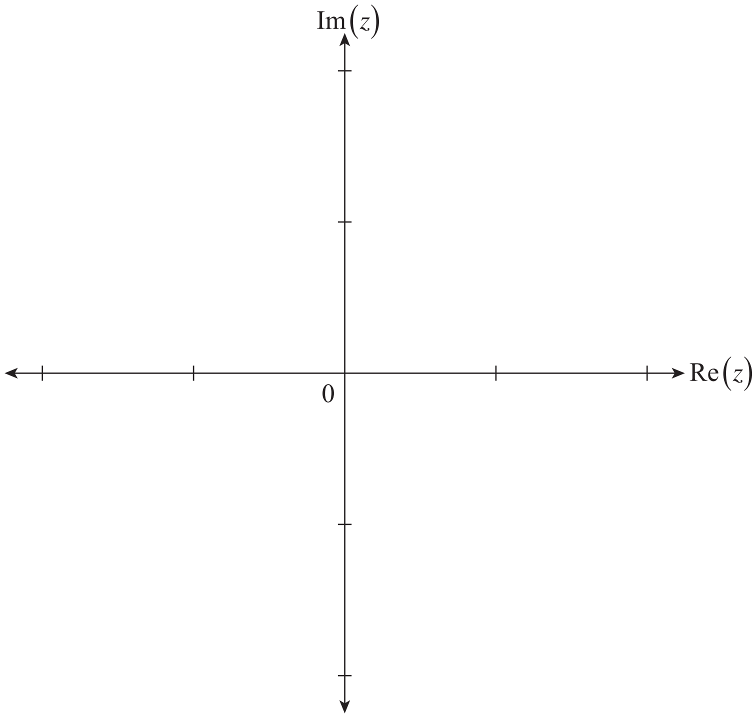


Figure 2 (5 marks)

(b) Find the *exact* Cartesian form of the complex number(s) in:

(i) the intersection of sets B and C.

(1 mark)

(ii) the intersection of sets A and B.

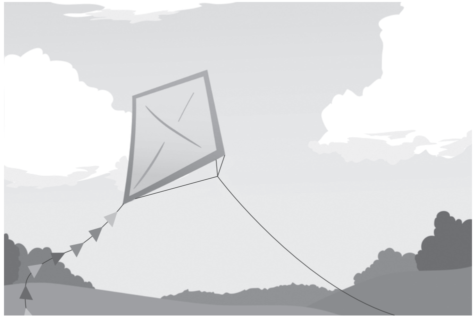
(2 marks)

Question 4 (8 marks)

Consider a kite that will fly above the ground in strong wind. The differential equation that models the height of the kite during its flight is given by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \left(1 + y^2\right)e^{-\frac{1}{2}t},\ t \geq 1$$

where y is the height of the kite above the ground (in metres) and $y \geq 0$. The time of flight, t, is measured in minutes.



Source: © adapted from Alexeyzet | Dreamstime.com

Figure 3 shows the slope field of the solutions to the differential equation.

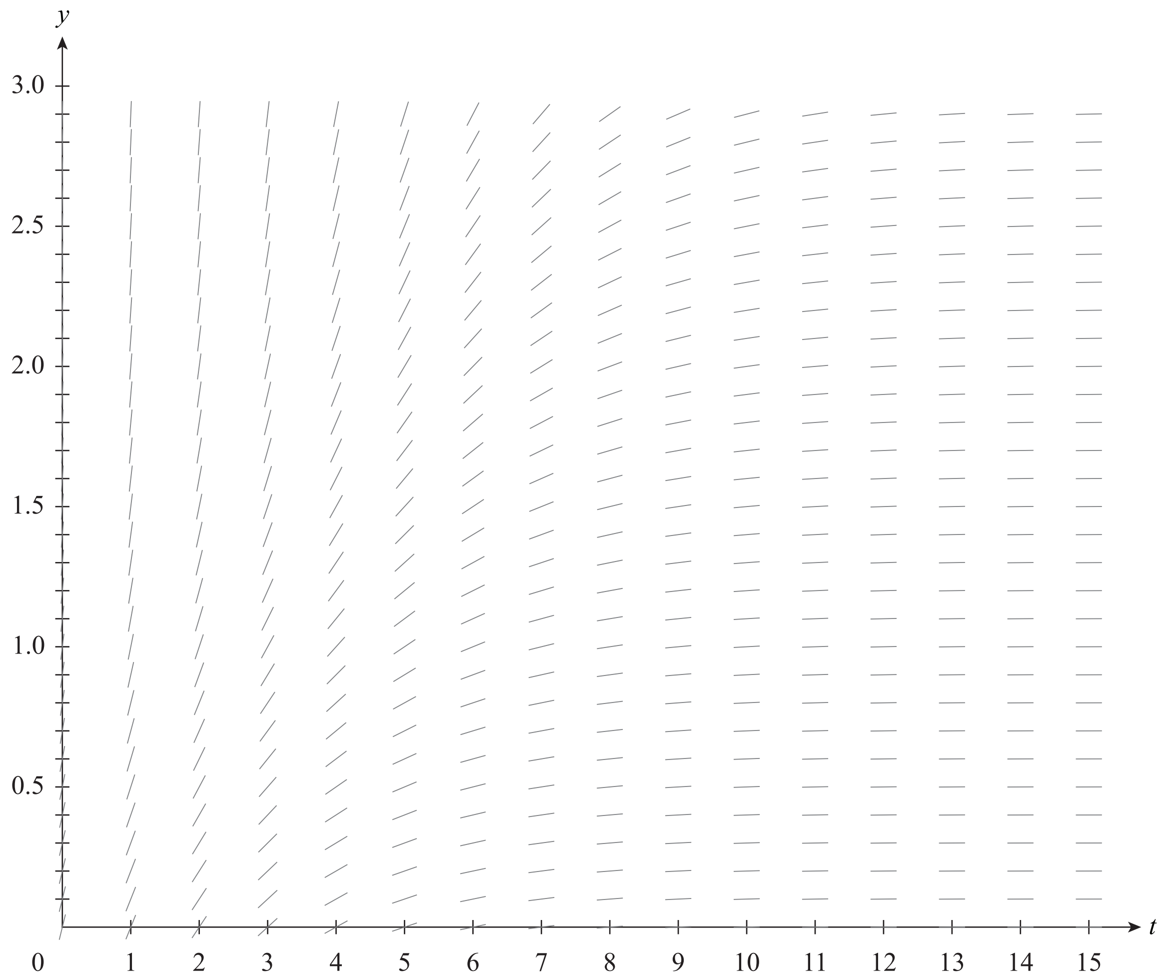


Figure 3

(a) On the slope field in Figure 3, draw the solution curve for the differential equation if at time $t = 1$, $y = 0$.

(2 marks)

(b) (i) Using integration, show that the solution curve with the condition $y = 0$ when $t = 1$ is given by:

$$y = \tan\left(-2e^{-\frac{1}{2}t} + 2e^{-\frac{1}{2}}\right).$$

(4 marks)

(ii) Calculate the height of the kite at $t = 10$.

(1 mark)

(iii) What is the maximum height that the kite approaches between $t = 1$ and $t = 15$, correct to three significant figures?

(1 mark)

Question 5 (8 marks)

(a) Using mathematical induction, prove that $7^n + 3n + 8$ is divisible by 9 for all positive integers n.

(6 marks)

(b) It was proved in part (a) that for all positive integers n, $7^n + 3n + 8 = 9A$, where A is an integer.

Hence show that $7^{2021} + 5$ is divisible by 3.

(2 marks)

Question 6 (7 marks)

Let $f(x)=\dfrac{1}{\sqrt{x}}-1$ and $g(x)=4-x^2$.

(a) (i) Show that $f(g(x))=\dfrac{1}{\sqrt{(2-x)(2+x)}}-1$.

(1 mark)

(ii) For what values of x is $f(g(x))$ defined?

(1 mark)

(iii) State the *exact* values of the x-intercepts of the graph of $y=f(g(x))$, in the form of $x=\pm a$.

(1 mark)

A sketch of the graph of $y=f(g(x))$ is shown in Figure 4.

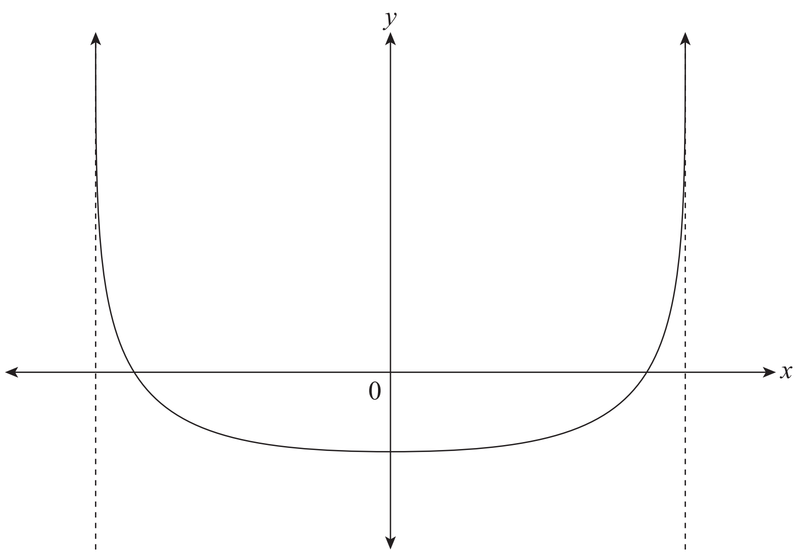


Figure 4

(b) Show that $\left(\frac{1}{\sqrt{(2-x)(2+x)}}-1\right)^2=\frac{1}{4-x^2}+1-\frac{2}{\sqrt{4-x^2}}$.

(1 mark)

(c) A solid is obtained when the region of the graph of $y=f\left(g\left(x\right)\right)$ that is bounded by the lines $x=\pm a$ is rotated about the x-axis.

(i) Show that the volume, V, of the solid is given by the equation below.

$$V=2\pi\int_0^a\left(1+\frac{1}{4-x^2}-\frac{2}{\sqrt{4-x^2}}\right)\mathrm{d}x$$

(2 marks)

(ii) Using your answer to part (a)(iii), find the volume of the solid.

(1 mark)

Question 7 (11 marks)

(a) (i) On the axes in Figure 5, sketch the graph of $f(x) = x\sin\left(\frac{x}{3}\right)$ for $-\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$.

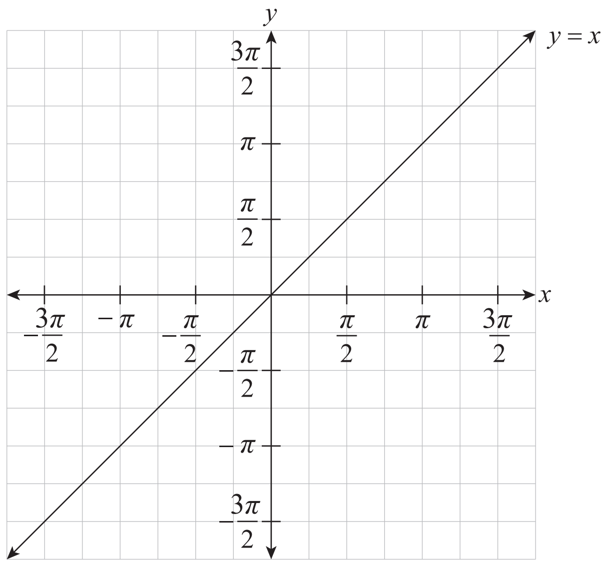


Figure 5

(3 marks)

(ii) Let $g(x) = x\sin\left(\frac{x}{3}\right)$ for $0 \le x \le \frac{3\pi}{2}$.

Explain why $g(x)$ has an inverse function, $g^{-1}(x)$.

(1 mark)

(iii) On the axes in Figure 5, sketch the graph of $g^{-1}(x)$, labelling it clearly.

*Do **not** attempt to find the formula for $g^{-1}(x)$.*

(1 mark)

(b) (i) Using integration by parts, show that $\int x\sin\left(\frac{x}{3}\right)\mathrm{d}x = -3x\cos\left(\frac{x}{3}\right) + 9\sin\left(\frac{x}{3}\right) + c$, where c is a constant.

(3 marks)

(ii) Find the *exact* area enclosed between the graphs of $f(x)$ and $g^{-1}(x)$ on the interval $0 \le x \le \frac{3\pi}{2}$.

(3 marks)

Question 8 (14 marks)

Consider the points $A(5, 2, 3)$ and $B(1, -2, 11)$, and the plane P_1 that is defined by the equation $x + y - 2z = -11$.

(a) (i) Find $\overrightarrow{BA}$.

(1 mark)

(ii) Points A and B are on the line l_1.

Show that l_1 is defined by the following parametric equations:

$$\begin{cases} x = 5 + t \\ y = 2 + t \\ z = 3 - 2t \end{cases} \quad \text{where } t \text{ is a real parameter.}$$

(2 marks)

(iii) Show that l_1 is perpendicular to P_1.

(1 mark)

(b) Let X be the point of intersection of l_1 and P_1 as shown in Figure 6.

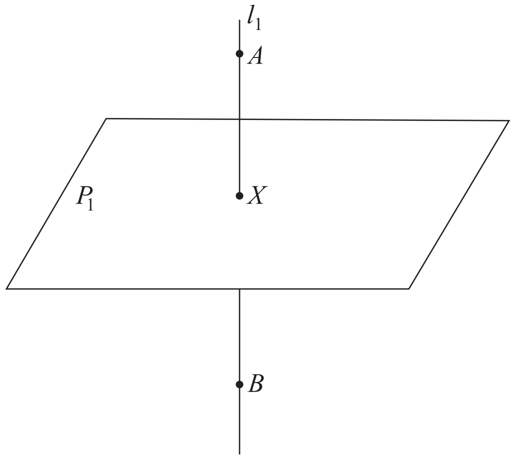


Figure 6

(i) Show that X has coordinates $(3, 0, 7)$.

(2 marks)

(ii) Show that $\overrightarrow{BX} = \overrightarrow{XA}$.

(1 mark)

(c) The plane P_2 is defined by the equation $2x - y + z = 15$.

Show that B is on P_2.

(1 mark)

(d) Points A and C are on opposite sides of P_2, as shown in Figure 7.

The line through A and C is normal to P_2.

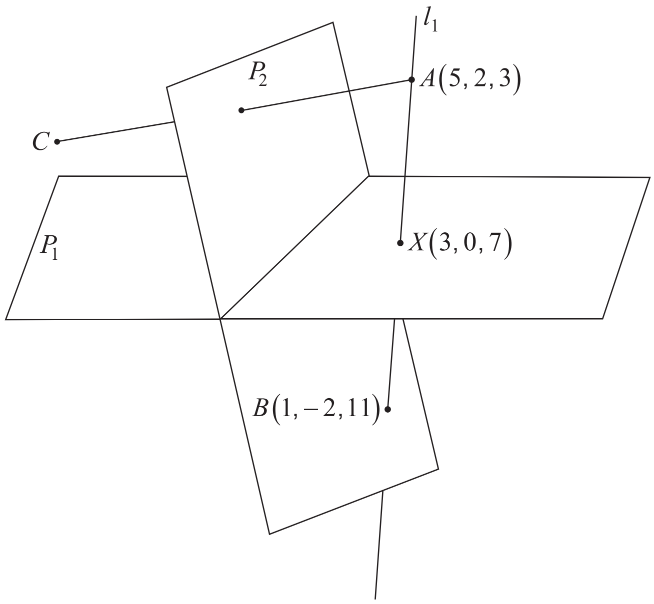


Figure 7

(i) Given that A and C are the same distance from P_2, find the coordinates of C.

(3 marks)

(ii) Is C closer to P_1 or to P_2? Explain your answer.

(3 marks)

Question 9 (16 marks)

The position of a particle at time t is described by the equations

$$\begin{cases} x(t) = e^{-t}(\cos t + \sin t) \\ y(t) = e^{-t}\sin 2t \end{cases} \quad \text{where } 0 \le t \le 2\pi.$$

(a) (i) Find the initial position of the particle.

(1 mark)

(ii) Find the *exact* position of the particle at $t = 2\pi$.

(1 mark)

(b) Show that the velocity vector of the particle is given by

$$\boldsymbol{v} = \left[-2e^{-t}\sin t, e^{-t}(2\cos 2t - \sin 2t)\right].$$

(3 marks)

(c) (i) Show that $\dfrac{dy}{dx} = \dfrac{\sin 2t - 2\cos 2t}{2\sin t}$ when $\sin t \neq 0$.

(2 marks)

(ii) When $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, show that $t=\frac{1}{2}\arctan(2)+\frac{k\pi}{2}$, where k is an integer.

(2 marks)

(iii) When $k=0$, $t=t_1$. When $k=1$, $t=t_2$.

Find the *exact* value of t_1 and of t_2.

(1 mark)

Figure 8 shows the graph of the position of the particle for $0 \le t \le 2\pi$.

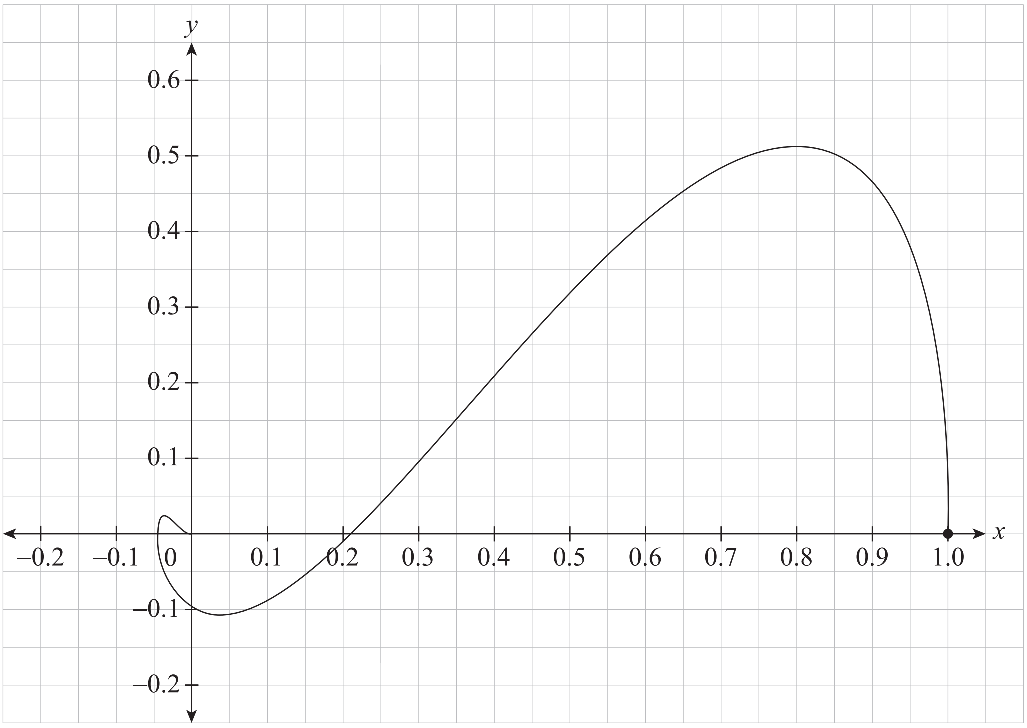


Figure 8

(d) On the graph in Figure 8, clearly indicate and label the position of the particle at t_1 and at t_2.

(1 mark)

(e) (i) Show that the speed of the particle is given by $S = e^{-t}\sqrt{4\sin^2 t + \left(2\cos 2t - \sin 2t\right)^2}$.

(2 marks)

(ii) (1) Find the length of the path travelled by the particle between $t = 0$ and $t = t_1$.

(2 marks)

(2) Find the length of the path travelled by the particle between $t = t_1$ and $t = t_2$.

(1 mark)

Question 10 (15 marks)

(a) (i) Write in Cartesian form all the solutions of the equation $z^4 = 1$, where z is complex.

(2 marks)

(ii) By factorising $z^4 - 1$, find all the solutions of the equation $z^3 - z^2 + z - 1 = 0$.

(2 marks)

Let $q(z) = z^2 + bz + c$, where b and c are real constants.

One zero of $q(z)$ is $r\operatorname{cis}\theta$, where $0 < r \le 1$ and $0 < \theta < \frac{\pi}{2}$.

(b) (i) State the other zero of $q(z)$, in terms of r and θ.

(1 mark)

(ii) Find $q(z)$ in expanded form, with coefficients in terms of r and θ.

(2 marks)

(c) Let $p(z)=\left(z^3-z^2+z-1\right)q(z)$.

(i) On the Argand diagram in Figure 9:

(1) plot all the zeros of $p(z)$. (2 marks)

(2) label the zeros w_1, w_2, w_3, w_4, and w_5 such that $\operatorname{Im} w_1 > \operatorname{Im} w_2 > \operatorname{Im} w_3 > \operatorname{Im} w_4 > \operatorname{Im} w_5$. (1 mark)

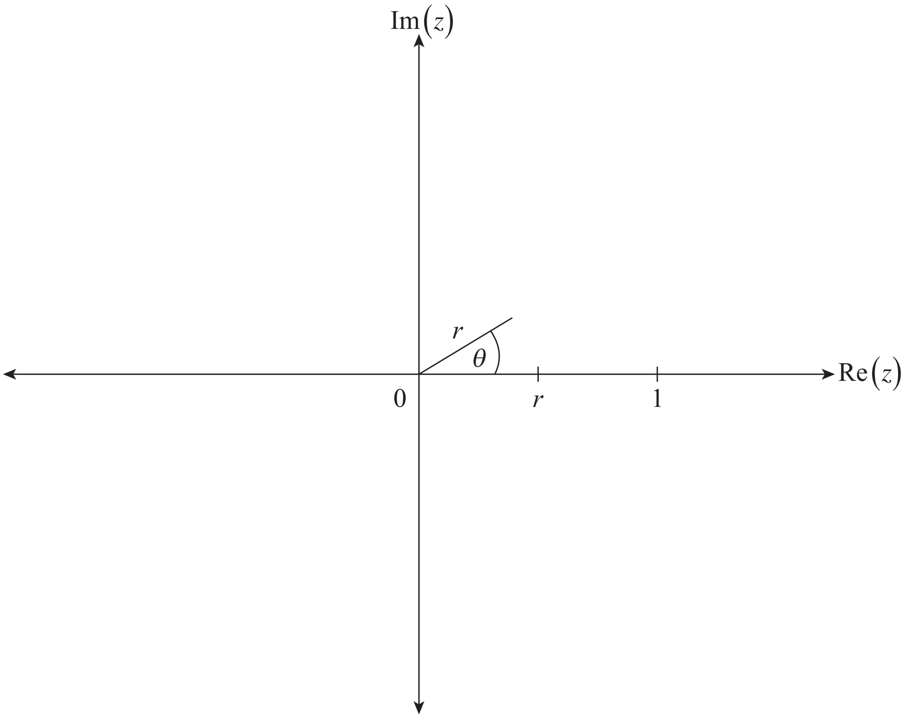


Figure 9

(ii) Explain why $|w_1-w_2|+|w_2-w_3|+|w_3-w_4|+|w_4-w_5| \geq 2\sqrt{2}$.

(2 marks)

(iii) Consider $p(z)$ when $\theta = \frac{\pi}{4}$, as shown in Figure 10.

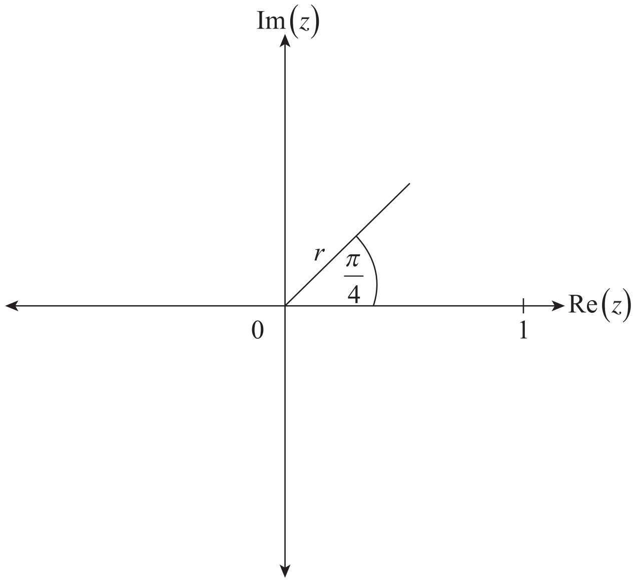


Figure 10

Find the value of r for which $|w_1 - w_2| + |w_2 - w_3| + |w_3 - w_4| + |w_4 - w_5| = 2\sqrt{2}$.

(1 mark)

(iv) Consider $p(z)$ when $r=1$ and $0<\theta<\frac{\pi}{2}$.

Using Figure 11, explain why $|w_1-w_2|+|w_2-w_3|+|w_3-w_4|+|w_4-w_5|<\pi$.

Note: It may be helpful to plot the zeros of $p(z)$ on Figure 11.

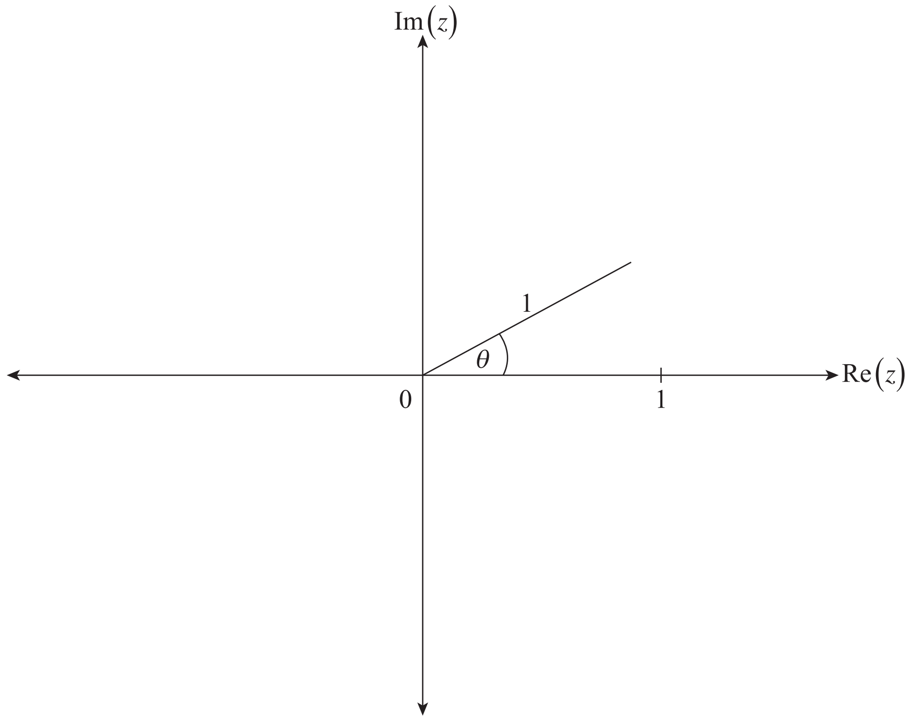


Figure 11

(2 marks)